The Berggren basis in elastic scattering

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PHY982 Project

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Let's set up the problem

- 2 particles in relative frame
- short-ranged interaction
- spherical symmetry

With partial wave decomposition:

$$\left(-\frac{1}{2\mu}\frac{d^2}{dR^2}+\frac{l(l+1)}{2\mu R^2}+V(R)\right)u(R)=Eu(R)$$

What kinds of solutions would you expect?

There are bound states

- $u \to 0$
- discrete spectrum
- normalizable
- imaginary momenta

(of course, they only exist if the interaction is sufficiently attractive)

and scattering states

(a.k.a. "continuum states")

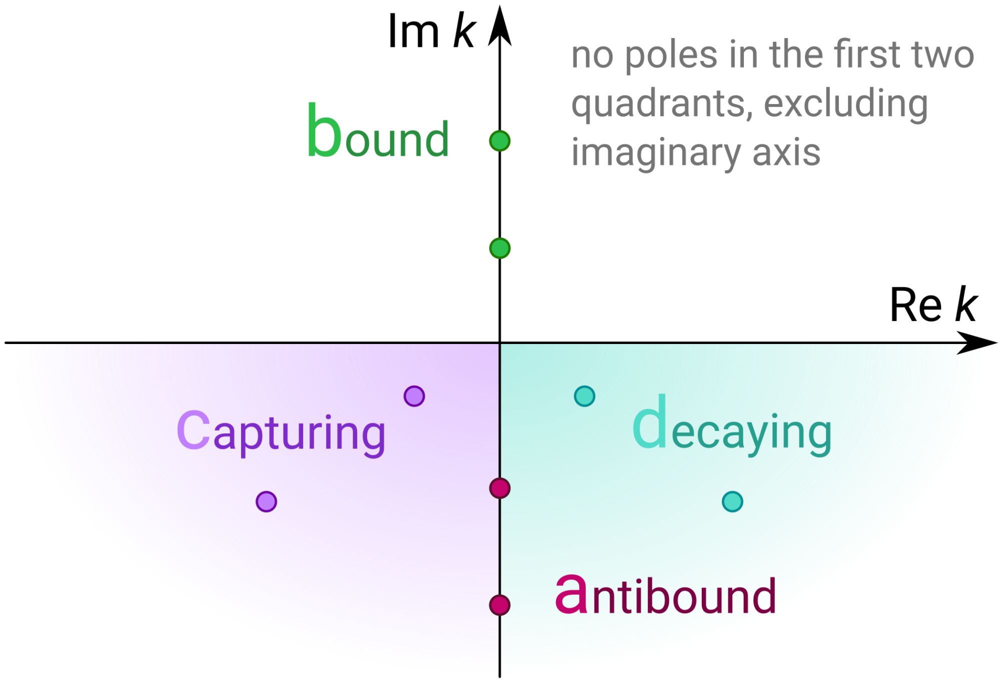
one of the main topics of this class

- $u \to e^{\pm ikx}$
- continuous spectrum
- *not* normalizable
- real momenta

but also resonance states

- $u \to e^{i(\pm k - i\kappa)x}$
- discrete spectrum
- not normalizable
- complex momenta
- **finite lifetime**

which correspond to poles in the S-matrix



How do they relate to each other?

*Newton completeness relation**

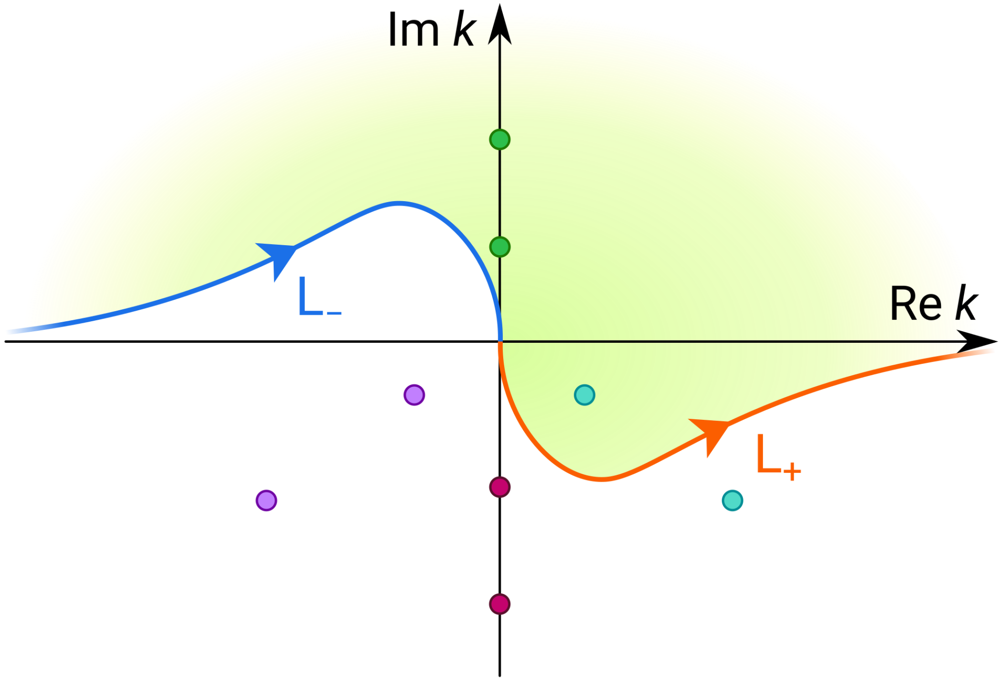
any state is a linear combination of bound and scattering states

$$\sum_{k\in\mathrm{bound}} |\varphi_k\rangle\langle\varphi_k| + \int_0^\infty k^2 dk\, |\varphi(k)\rangle\langle\varphi(k)| = \hat{1}$$

but what about resonance states?

*R. G. Newton, *Scattering Theory of Waves and Particles*

What if we deform the contour?



We then obtain the *Berggren completeness relation**

$$\sum_{k \in \text{enclosed b+d}} |\varphi_k\rangle\langle\varphi_k| + \int_{\mathrm{L}_+} k^2 dk \, |\varphi(k)\rangle\langle\varphi(k)| = \hat{1}$$

The discrete sum is over all the poles enclosed by the contour on the upper half plane, which includes bound and resonance states.

*T. Berggren, *Nucl. Phys. A* **109** (1968) 265.

and hence the *Berggren basis*

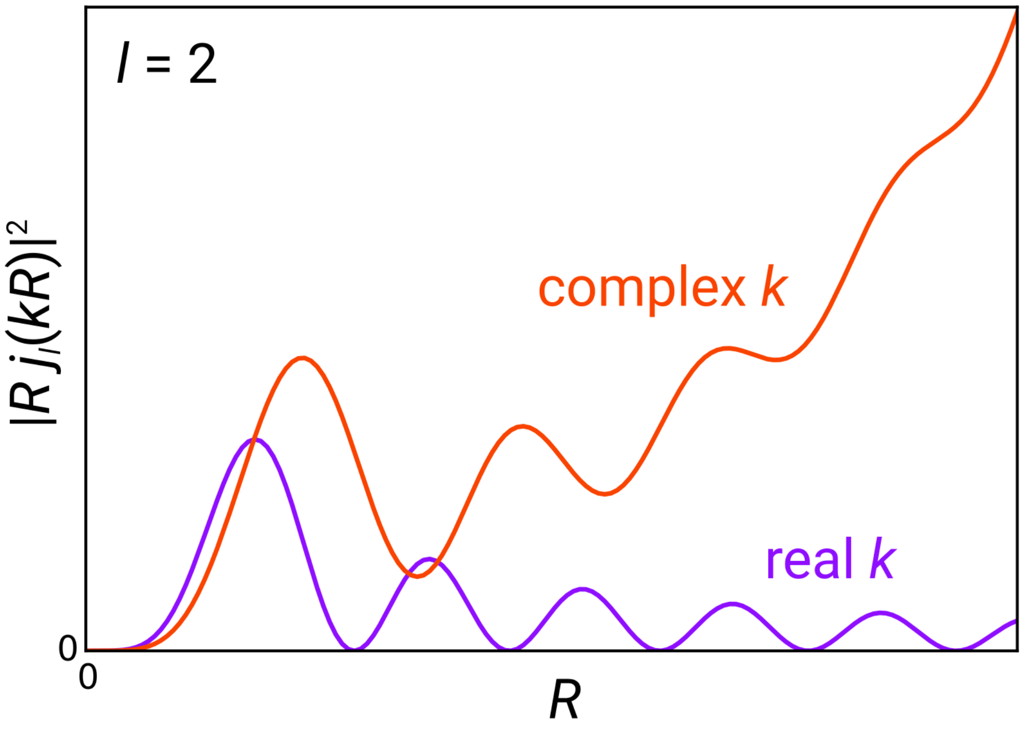
What do we get?

- bound, scattering, *and resonance* states

What's the price?

- complex energies; H is no longer hermitian!
- integral divergence (worse than scattering)

Example in free space



But it's not that bad

- formally, can use nonhermitian quantum mechanics with a rigged Hilbert space (not as crazy as it sounds)
- regularization techniques to fix the integral:
  - Gaussian regularization ($e^{-\varepsilon R^2}$, then take limit as $\varepsilon \to 0$)
  - complex scaling (rotate the contour until it becomes well-behaved)

So about that scattering problem...

Let's find the Berggren states of the ^{11}Be system studied in homework 1.

We will use **basis expansion** and we will solve this in momentum space (which is quite natural for Berggren states).

Here's the momentum-space Schrödinger equation

(can be derived by taking a Fourier transform)

$$\frac{k^2}{2\mu}\varphi(k) + \int_0^\infty \kappa^2 d\kappa\, V(k,\kappa)\varphi(\kappa) = E\varphi(k)$$

where $V(k,\kappa) \equiv \frac{2}{\pi}\int_0^\infty R^2 dR\, j_l(kR)V(R)j_l(\kappa R)$

Then we discretize the continuum

$$\frac{k^2}{2\mu}\varphi_k + \sum_\kappa \kappa^2 w_\kappa V_{k\kappa}\varphi_\kappa = E\varphi_k$$

Points may be chosen via a quadrature scheme with weights w_κ.

We'll use Gauss-Legendre here.

and obtain an eigenvalue problem

which we can plug into Lapack or something

$$\sum_{\kappa} H_{k\kappa}\varphi_{\kappa} = E\varphi_{k}$$

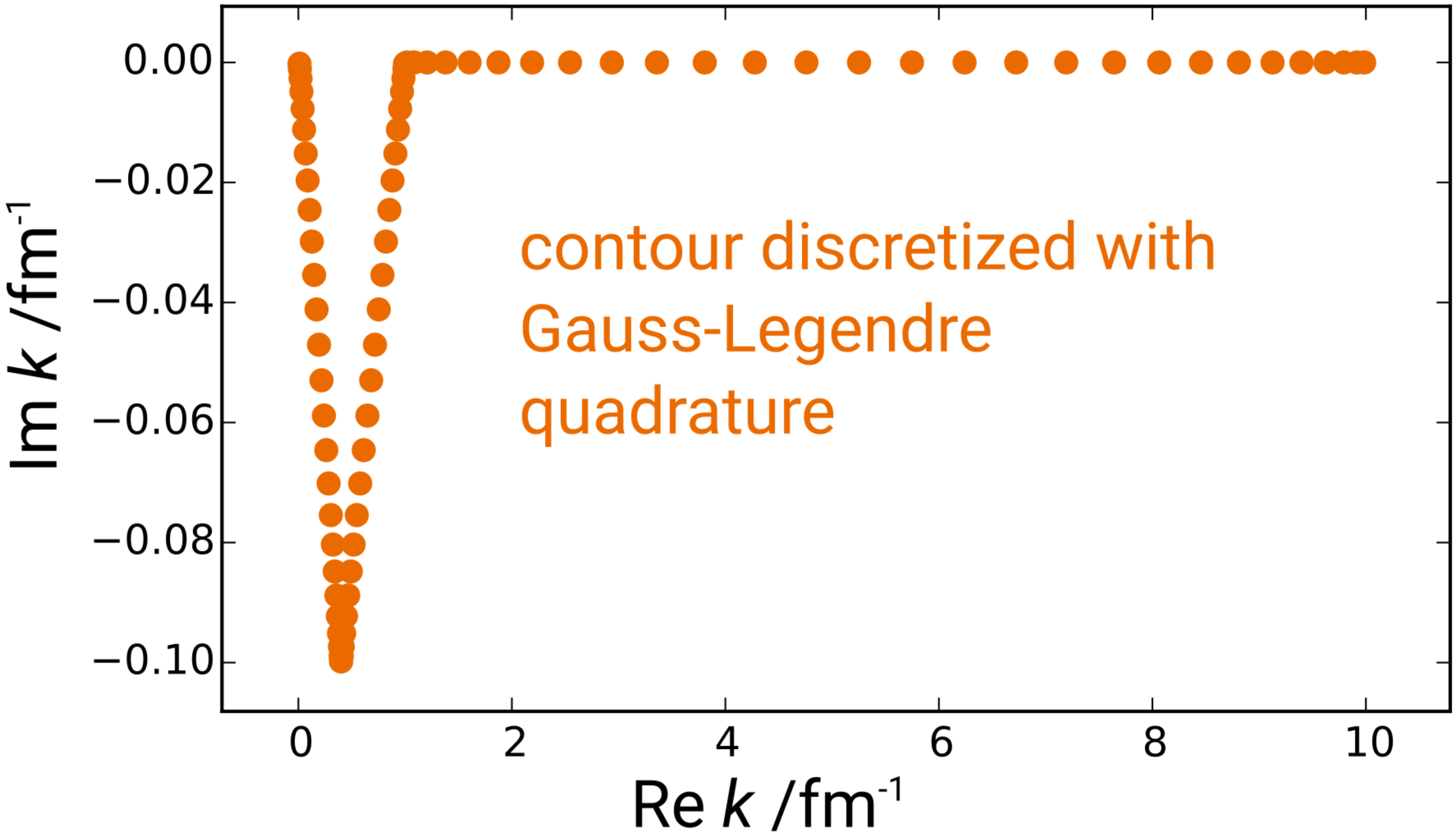
where $H_{k\kappa} = \frac{k^2}{2\mu}\delta_{k\kappa} + \kappa^2 w_{\kappa} V_{k\kappa}$

How do we recover the position-space wavefunction?

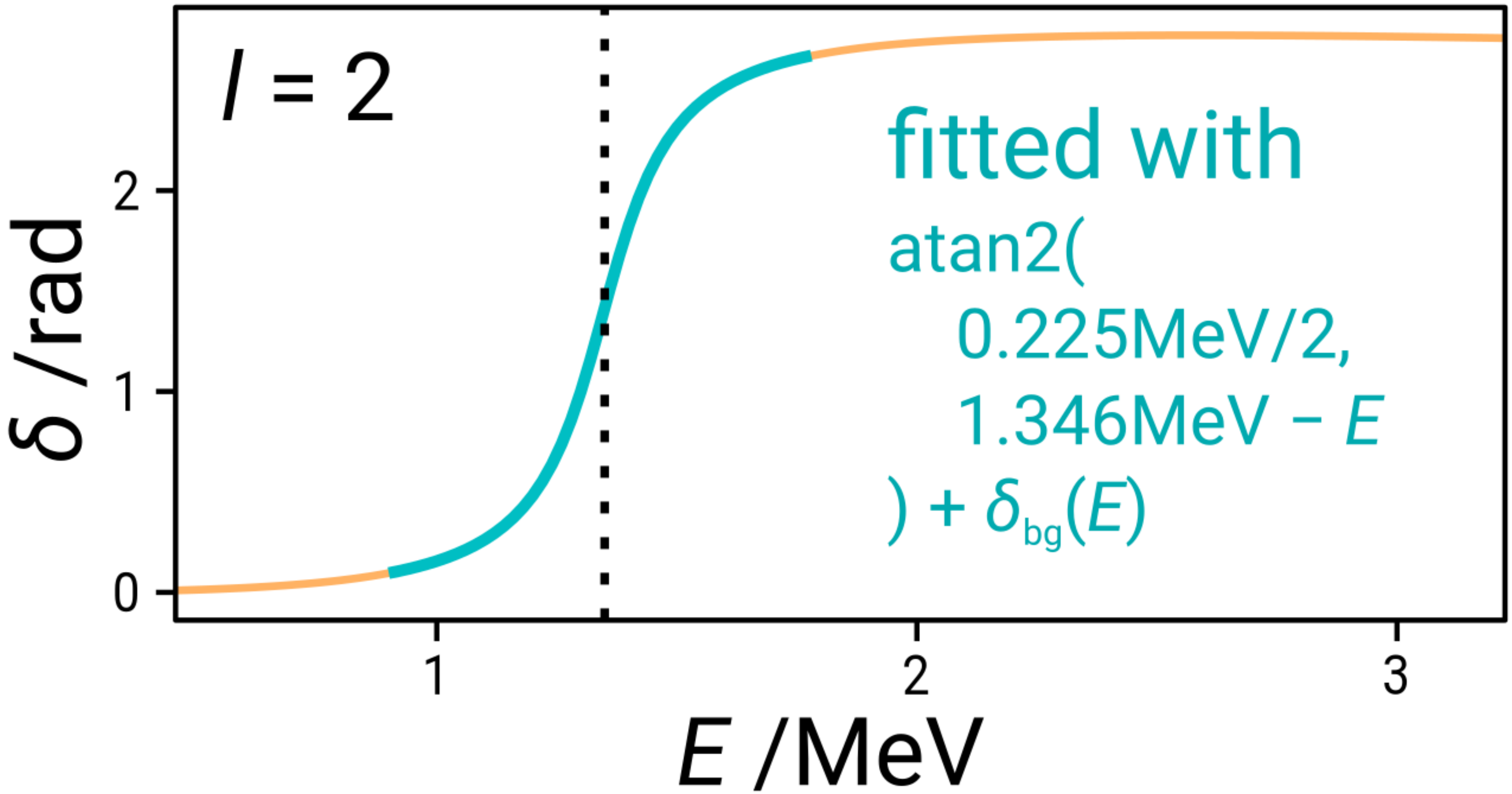
Just do a Hankel transform:

$$u(R) = i^l \sqrt{\frac{2}{\pi}} R \sum_k k^2 w_k j_l(kR) \varphi_k$$

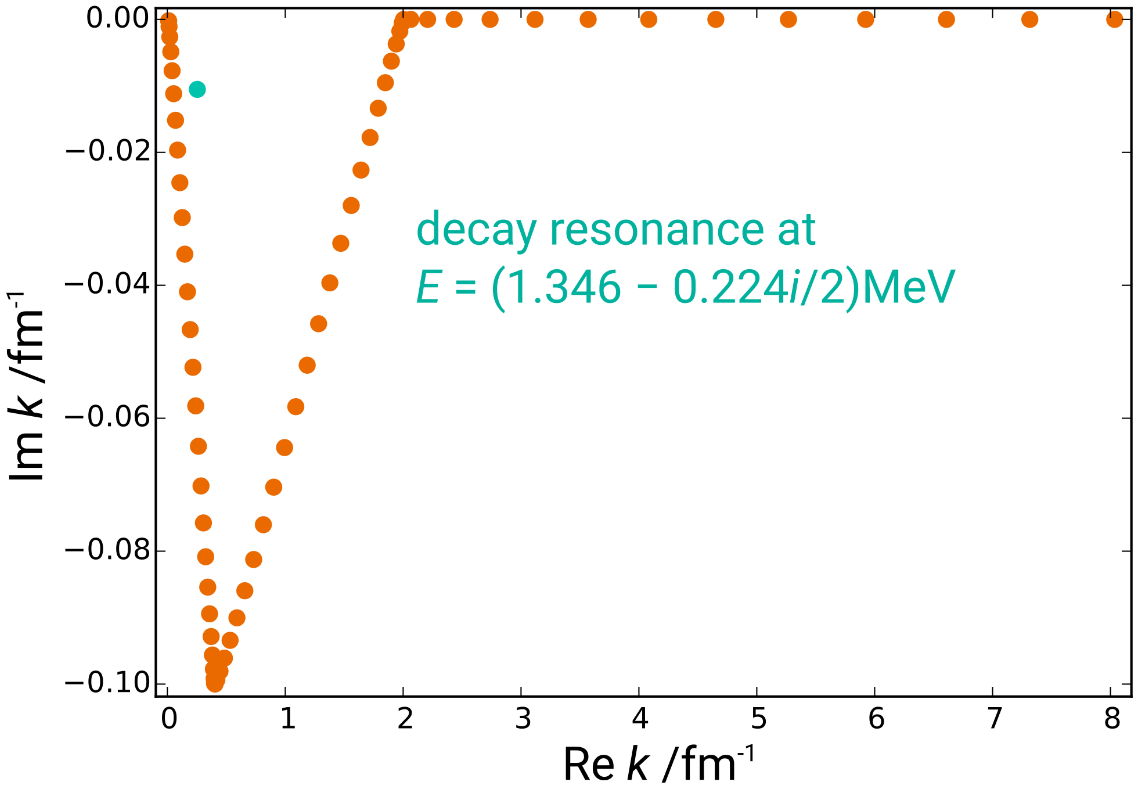
What does the contour look like?



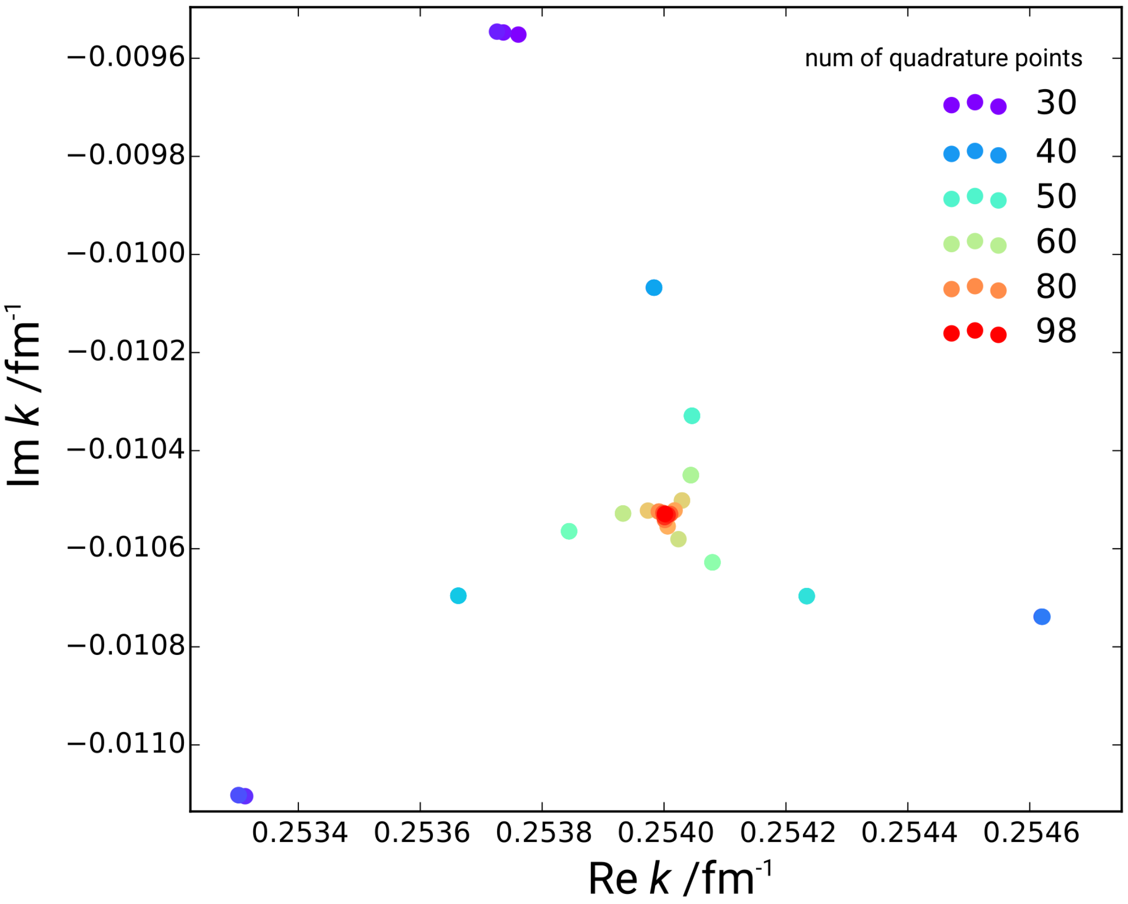
Recall from our first homework



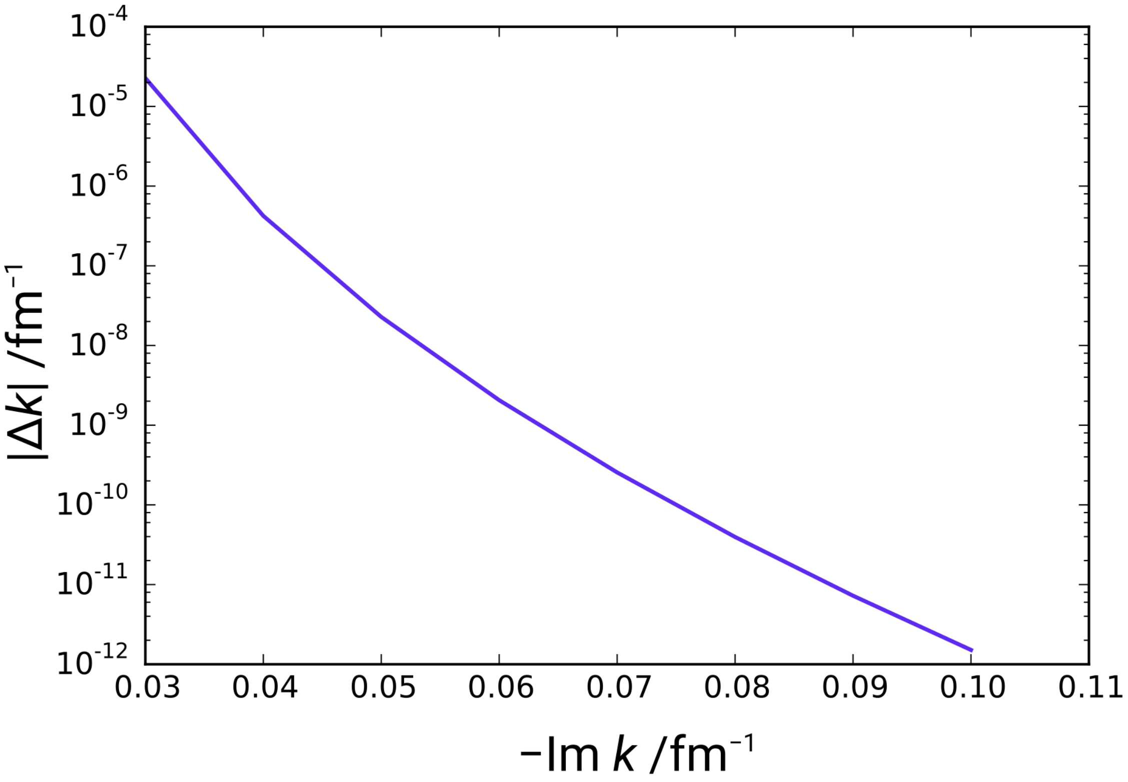
Do we get the right answer?



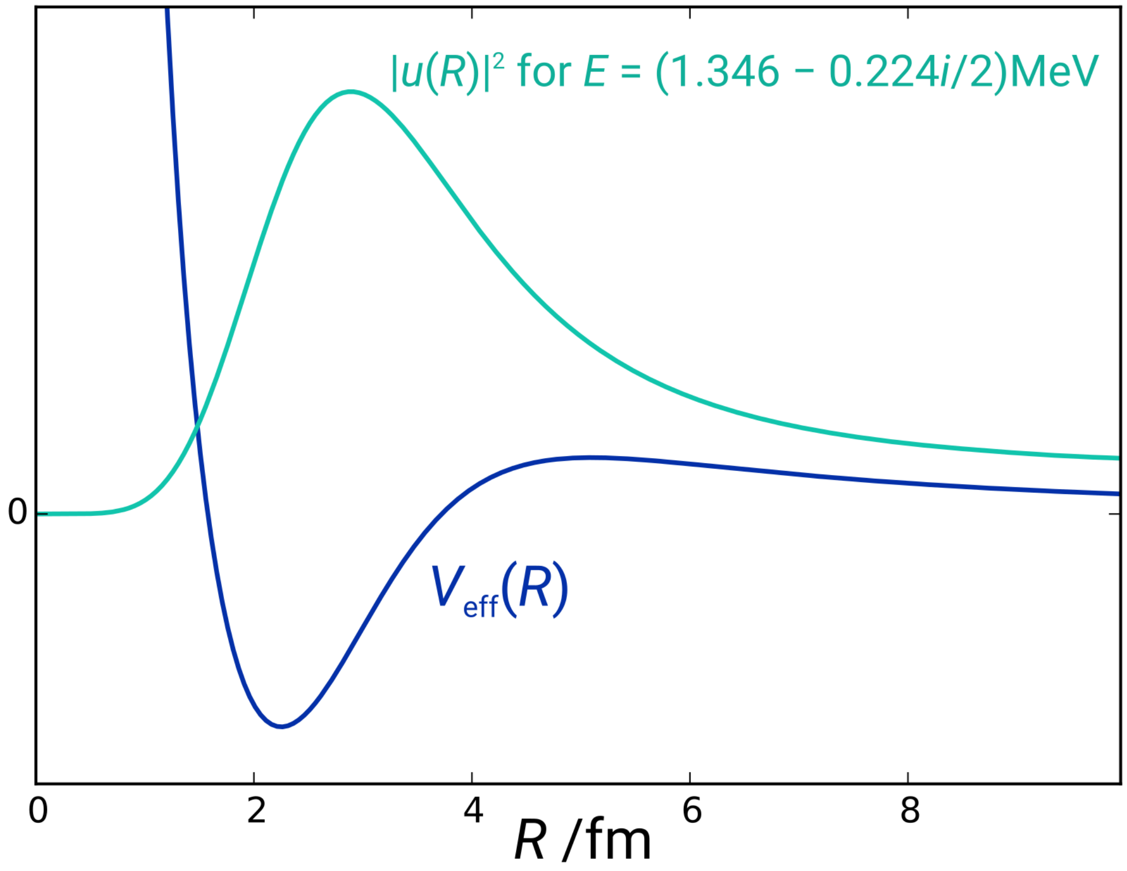
How well does it converge? (I)



How well does it converge? (II)

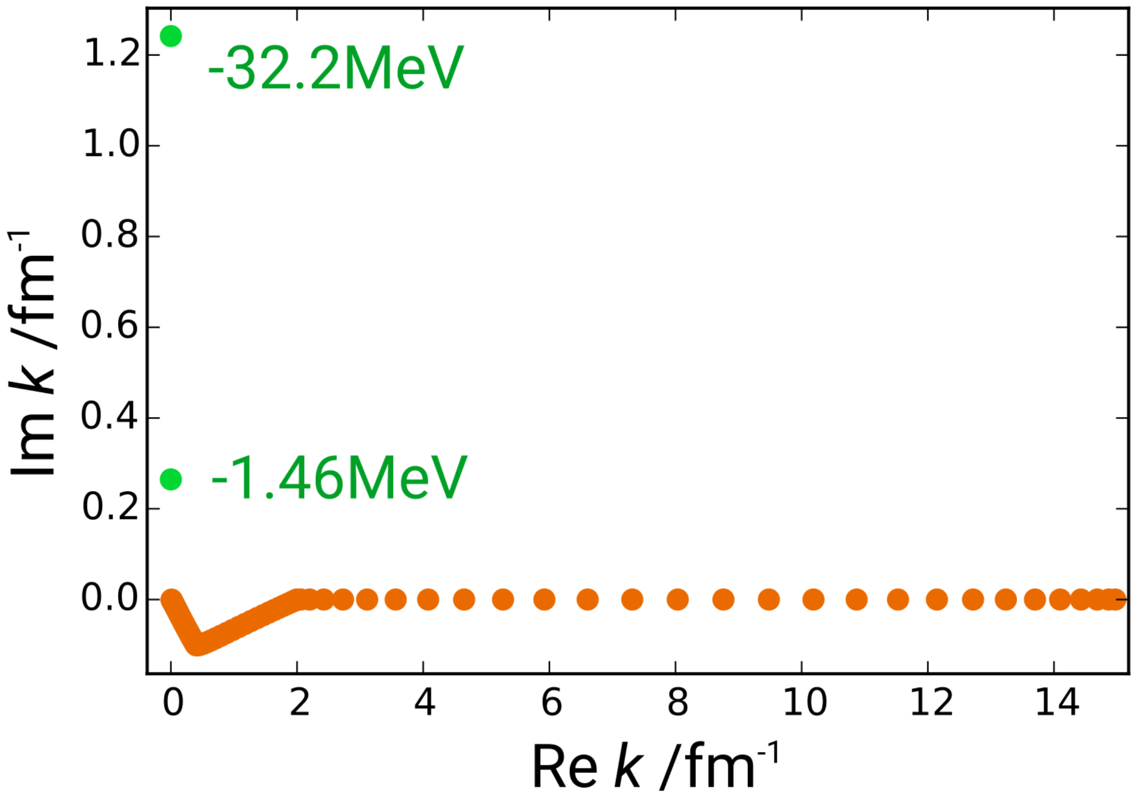


What does the state look like?

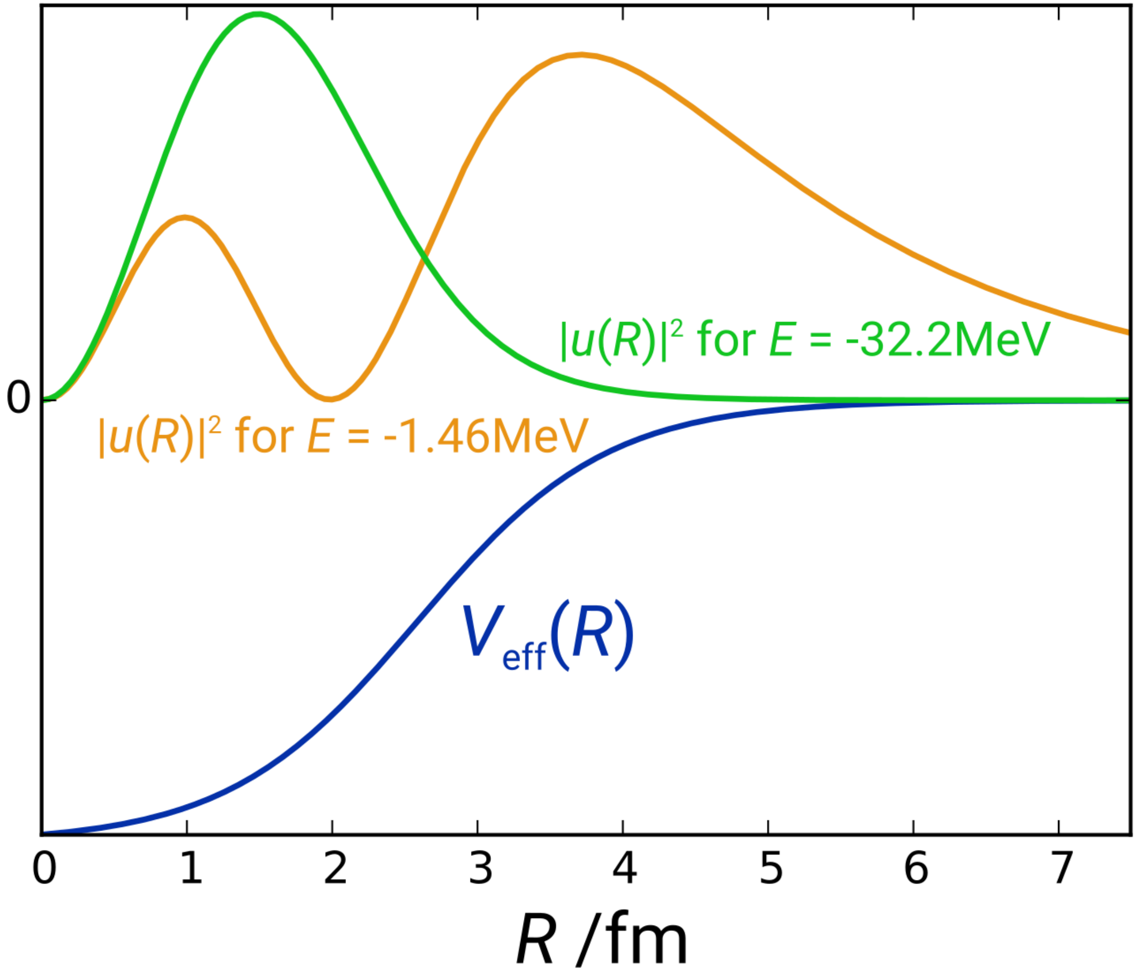


There are bound states too!

(in $l = 0$)



What do those look like?



Unanswered questions…

- *How do the states behave asymptotically?* Accurate asymptotics seems to be hard to obtain in this approach. Might require a lot more points.
- *How can we do this more efficiently?* Calculating the momentum matrix elements of V is the dominant expense. It gets worse as the momentum increases: the integrals become extremely oscillatory.

Thanks for listening!

https://github.com/xrf/phy982-proj